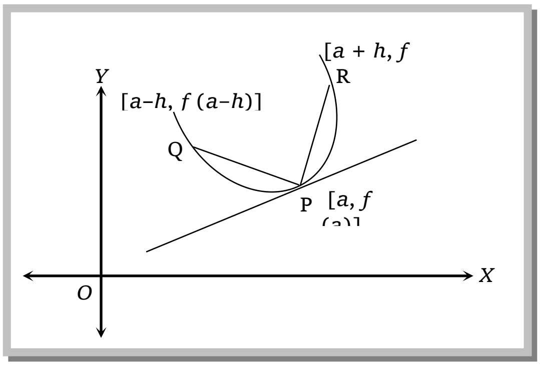
2.4 Differentiability

2.4.1 Differentiability of a Function at a Point

(1) **Meaning of differentiability at a point :** Consider the function $f(x)$ defined on an open interval (b, c) let $P(a, f(a))$ be a point on the curve $y = f(x)$ and let $Q(a-h, f(a-h))$ and $R(a+h, f(a+h))$ be two neighbouring points on the left hand side and right hand side respectively of the point P.



Then slope of chord $PQ = \dfrac{f(a-h)-f(a)}{(a-h)-a} = \dfrac{f(a-h)-f(a)}{-h}$

and, slope of chord $PR = \dfrac{f(a+h)-f(a)}{a+h-a} = \dfrac{f(a+h)-f(a)}{h}$.

$\because$ As $h \to 0$, point Q and R both tends to P from left hand and right hand respectively. Consequently, chords PQ and PR becomes tangent at point P.

Thus, $\lim\limits_{h\to 0} \dfrac{f(a-h)-f(a)}{-h} = \lim\limits_{h\to 0}$ (slope of chord PQ)= $\lim\limits_{Q\to P}$ (slope of chord PQ)

Slope of the tangent at point P, which is limiting position of the chords drawn on the left hand side of point P and $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = \lim\limits_{h\to 0}$ (slope of chord PR) $= \lim\limits_{R\to P}$ (slope of chord PR).

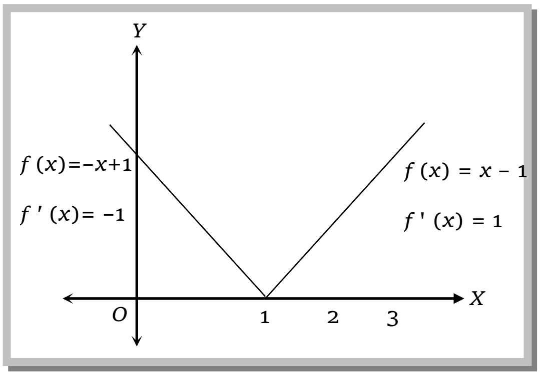
$\Rightarrow$ Slope of the tangent at point P, which is the limiting position of the chords drawn on the right hand side of point P.

Now, $f(x)$ is differentiable at $x = a \Leftrightarrow \lim\limits_{h\to 0} \dfrac{f(a-h)-f(a)}{-h} = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

$\Leftrightarrow$ There is a unique tangent at point P.

Thus, $f(x)$ is differentiable at point P, iff there exists a unique tangent at point P. In other words, $f(x)$ is differentiable at a point P iff the curve does not have P as a corner point. *i.e.*, "the function is not differentiable at those points on which function has jumps (or holes) and sharp edges."

Let us consider the function $f(x) = |x-1|$, which can be graphically shown,



Which show $f(x)$ is not differentiable at $x = 1$. Since, $f(x)$ has sharp edge at $x = 1$.

Mathematically : The right hand derivative at $x=1$ is 1 and left-hand derivative at $x=1$ is –1. Thus, $f(x)$ is not differentiable at $x=1$.

(2) **Right hand derivative :** Right hand derivative of $f(x)$ at $x=a$, denoted by $f'(a+0)$ or $f'(a+)$, is the $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

(3) **Left hand derivative :** Left hand derivative of $f(x)$ at $x=a$, denoted by $f'(a-0)$ or $f'(a-)$, is the $\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$.

(4) A function $f(x)$ is said to be differentiable (finitely) at *x* = *a* if $f'(a+0)=f'(a-0)=$ finite

i.e., $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$ = finite and the common limit is called the derivative of $f(x)$ at $x=a$, denoted by $f'(a)$. Clearly, $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ {$x\to a$ from the left as well as from the right}.

Example: 1 Consider $f(x)=\begin{cases}\frac{x^2}{|x|}, & x\neq 0\\ 0, & x=0\end{cases}$ **[EAMCET 1994]**

(a) $f(x)$ is discontinuous everywhere

(b) $f(x)$ is continuous everywhere but not differentiable at *x* = 0

(c) $f'(x)$ exists in (–1, 1)

(d) $f'(x)$ exists in (–2, 2)

Solution: (b) We have, $f(x)=\begin{cases}\frac{x^2}{|x|}, & x\neq 0\\ 0, & x=0\end{cases}$ = $\begin{cases}\frac{x^2}{x}=x, & x>0\\ 0, & x=0\\ \frac{x^2}{-x}=-x, & x<0\end{cases}$

$\Rightarrow \lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}-x=0$, $\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}x=0$ and $f(0)=0$.

So $f(x)$ is continuous at $x=0$. Also $f(x)$ is continuous for all other values of *x*. Hence, $f(x)$ is everywhere continuous.

Also, $Rf'(0)=f'(0+0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h-0}=\lim_{h\to 0}\frac{h-0}{h}=1$

i.e. $Rf'(0)=1$ and $Lf'(0)=f'(0-0)=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}=\lim_{h\to 0}\frac{h}{-h}=-1$

i.e. $Lf'(0)=-1$ So, $Lf'(0)\neq Rf'(0)$ *i.e.*, $f(x)$ is not differentiable at $x=0$.

Example: 2 If the function *f* is defined by $f(x)=\frac{x}{1+|x|}$, then at what points *f* is differentiable

(a) Everywhere (b) Except at $x=\pm 1$ (c) Except at $x=0$ (d) Except at $x=0$ or ± 1

Solution: (a) We have, $f(x)=\frac{x}{1+|x|}=\begin{cases}\frac{x}{1+x}, & x>0\\ 0, & x=0\\ \frac{x}{1-x}, & x<0\end{cases}$; $Lf'(0)=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}=\lim_{h\to 0}\frac{\frac{-h}{1+h}-0}{-h}=1$

$$Rf'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \frac{\lim_{h\to 0}\frac{h}{1+h}-0}{h} = \lim_{h\to 0}\frac{1}{1+h} = 1$$

So, $Lf'(0) = Rf'(0) = 1$

So, $f(x)$ is differentiable at $x=0$; Also $f(x)$ is differentiable at all other points.

Hence, $f(x)$ is everywhere differentiable.

Example: 3 The value of the derivative of $|x-1|+|x-3|$ at $x=2$ is

(a) -2 (b) 0 (c) 2 (d) Not defined

Solution: (b) Let $f(x) = |x-1|+|x-3| = \begin{cases} -(x-1)-(x-3) & , x<1 \\ (x-1)-(x-3) & , 1\le x<3 \\ (x-1)+(x-3) & , x\ge 3 \end{cases} = \begin{cases} -2x+4 & , x<1 \\ 2 & , 1\le x<3 \\ 2x-4 & , x\ge 3 \end{cases}$

Since, $f(x) = 2$ for $1\le x<3$. Therefore $f'(x)=0$ for all $x\in(1,3)$.

Hence, $f'(x)=0$ at $x=2$.

Example: 4 The function f defined by $f(x) = \begin{cases} \frac{\sin x^2}{x}, & x\ne 0 \\ 0 & , x=0 \end{cases}$

(a) Continuous and derivable at $x=0$ (b) Neither continuous nor derivable at $x=0$

(c) Continuous but not derivable at $x=0$ (d) None of these

Solution: (a) We have, $\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sin x^2}{x} = \lim_{x\to 0}\left(\frac{\sin x^2}{x^2}\right)x = 1\times 0 = 0 = f(0)$

So, $f(x)$ is continuous at $x=0$, $f(x)$ is also derivable at $x=0$, because $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{\sin x^2}{x^2} = 1$ exists finitely.

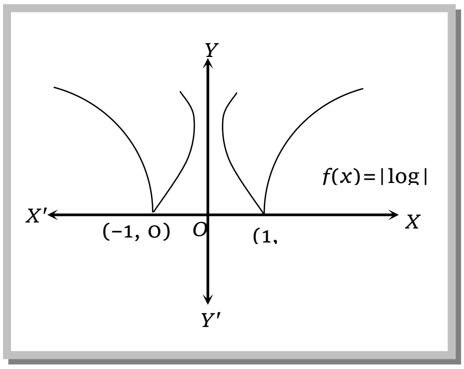
Example: 5 If $f(x) = |\log|x||$, then

(a) $f(x)$ is continuous and differentiable for all x in its domain

(b) $f(x)$ is continuous for all x in its domain but not differentiable at $x=\pm 1$.

(c) $f(x)$ is neither continuous nor differentiable at $x=\pm 1$

(d) None of these



Solution: (b) It is evident from the graph of $f(x) = |\log|x||$ that $f(x)$ is everywhere continuous but not differentiable at $x=\pm 1$..

Example: 6 The left hand derivative of $f(x) = [x]\ \sin(\pi x)$ at $x=k$ (k is an integer), is

(a) $(-1)^k(k-1)\pi$ (b) $(-1)^{k-1}(k-1)\pi$ (c) $(-1)^k k\pi$ (d) $(-1)^{k-1}k\pi$

Solution: (a) $f(x) = [x]\ \sin(\pi x)$

If x is just less than k, $[x] = k - 1$. $\therefore$ $f(x) = (k-1)\sin(\pi x)$, when $x<k$ $\forall k\in I$

Now L.H.D. at $x=k$,

$= \lim_{x \to k} \frac{(k-1)\sin(\pi x) - k\sin(\pi k)}{x-k} = \lim_{x \to k} \frac{(k-1)\sin(\pi x)}{(x-k)}$ [as $\sin(\pi k) = 0, k \in$ integer]

$= \lim_{h \to 0} \frac{(k-1)\sin(\pi(k-h))}{-h}$ [Let $x = (k-h)$]

$= \lim_{h \to 0} \frac{(k-1)(-1)^{k-1}\sin h\pi}{-h} = \lim_{h \to 0} (k-1)(-1)^{k-1}\frac{\sin h\pi}{h\pi} \times (-\pi) = (k-1)(-1)^k \pi = (-1)^k (k-1)\pi$.

Example: 7 The function $f(x) = |x| + |x-1|$ is

(a) Continuous at $x = 1$, but not differentiable (b) Both continuous and differentiable at $x = 1$

(c) Not continuous at $x = 1$ (d) None of these

Solution: (a) We have, $f(x) = |x| + |x-1| = \begin{cases} -2x+1, & x < 0 \\ 1, & 0 \le x < 1 \\ 2x-1, & x \ge 1 \end{cases}$

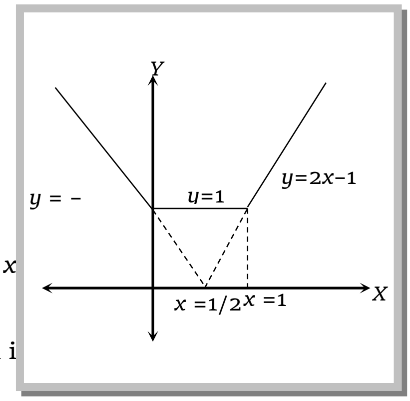
Since, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 1 = 1$, $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x-1) = 1$ and $f(1) = 2 \times 1 - 1 = 1$

$\therefore \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$. So, $f(x)$ is continuous at x = 1.

Now, $\lim_{x \to 1^-} \frac{f(x) - f(1)}{x-1} = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{1-1}{-h} = 0$,

and $\lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{2(1+h) - 1 - 1}{h} = 2$.

$\therefore$ (LHD at $x = 1$) $\ne$ (RHD at $x = 1$). So, $f(x)$ is not differentiable at x



Trick : The graph of $f(x) = |x| + |x-1|$ *i.e.* $f(x) = \begin{cases} -2x+1, & x < 0 \\ 1, & 0 \le x < 1 \\ 2x-1, & x \ge 1 \end{cases}$ i

By graph, it is clear that the function is not differentiable at $x = 0$, 1 as there it has sharp edges.

Example: 8 Let $f(x) = |x-1| + |x+1|$, then the function is

(a) Continuous (b) Differentiable except $x = \pm 1$

(c) Both (a) and (b) (d) None of these

Solution: (c) Here $f(x) = |x-1| + |x+1| \Rightarrow f(x) = \begin{cases} 2x, & \text{when } x > 1 \\ 2, & \text{when } -1 \le x \le 1 \\ -2x, & \text{when } x < -1 \end{cases}$

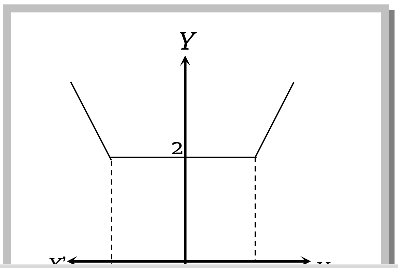
Graphical solution : The graph of the function is shown alongside,

From the graph it is clear that the function is continuous at all real x, also differentiable at all real x except at $x = \pm 1$; Since sharp edges at $x = -1$ and $x = 1$.

At $x = 1$ we see that the slope from the right *i.e.*, R.H.D. = 2, while slope from the left *i.e.*, L.H.D.= 0

Similarly, at $x = -1$ it is clear that R.H.D. = 0 while L.H.D. = – 2

Trick : In this method, first of all, we differentiate the function and on the derivative equality sign should be removed from doubtful points.



Here, $f'(x)=\begin{cases}-2 & , & x<-1\\ 0 & , & -1<x<1\\ 2 & , & x>1\end{cases}$ (No equality on –1 and +1)

Now, at $x=1,\ f'(1^+)=2$ while $f'(1^-)=0$ and

at $x=-1,\ f'(-1^+)=0$ while $f'(-1^-)=-2$

Thus, $f(x)$ is not differentiable at $x=\pm 1$.

Note : ❒ This method is not applicable when function is discontinuous.

Example: 9 If the derivative of the function $f(x)=\begin{cases}ax^2+b & , & x<-1\\ bx^2+ax+4 & , & x\geq -1\end{cases}$ is everywhere continuous and differentiable at x = 1 then

(a) a = 2, b = 3 (b) a = 3, b = 2 (c) a = –2, b = – 3 (d) a = – 3, b = – 2

Solution: (a) $f(x)=\begin{cases}ax^2+b & , & x<-1\\ bx^2+ax+4 & , & x\geq -1\end{cases}$

$\therefore f'(x)=\begin{cases}2ax & , & x<-1\\ 2bx+a & , & x\geq -1\end{cases}$

To find a, b we must have two equations in a, b

Since $f(x)$ is differentiable, it must be continuous at $x=-1$.

$\therefore\ R=L=V$ at $x=-1$ for $f(x)$ $\Rightarrow b-a+4=a+b$

$\therefore 2a=4$ *i.e.,* $a=2$

Again $f'(x)$ is continuous, it must be continuous at $x=-1$.

$\therefore R=L=V$ at $x=-1$ for $f'(x)$

$-2b+a=-2a.$ Putting $a=2,$ we get $-2b+2=-4$

$\therefore 2b=6$ or $b=3.$

Example: 10 Let f be twice differentiable function such that $f''(x)=-f(x)$ and $f'(x)=g(x),\ h(x)=\{f(x)\}^2+\{g(x)\}^2$. If $h(5)=11,$ then $h(10)$ is equal to

(a) 22 (b) 11 (c) 0 (d) None of these

Solution: (b) Differentiating the given relation $h(x)=[f(x)]^2+[g(x)]^2$ w.r.t x, we get $h'(x)=2f(x)f'(x)+2g(x)g'(x)$ ....(i)

But we are given $f''(x)=-f(x)$ and $f'(x)=g(x)$ so that $f''(x)=g'(x).$

Then (1) may be re-written as $h'(x)=-2f''(x)f'(x)+2f'(x)f''(x)=0.$ Thus $h'(x)=0$

Whence by intergrating, we get $h(x)=$ constant = c (say). Hence $h(x)=c,$ for all x.

In particular, $h(5)=c.$ But we are given $h(5)=11.$

It follows that $c=11$ and we have $h(x)=11$ for all x. Therefore, h(10) = 11.

Example: 11 The function $f(x)=\begin{cases}|2x-3|[x] & , & x\geq 1\\ \sin\left(\dfrac{\pi x}{2}\right) & , & x<1\end{cases}$

(a) Is continuous at $x=2$ (b) Is differentiable at $x=1$

(c) Is continuous but not differentiable at $x=1$ (d) None of these

Solution: (c) $[2+h]=2,[2-h]=1,[1+h]=1,[1-h]=0$

At $x = 2$, we will check $R = L = V$

$R = \lim_{h \to 0} |\, 4 + 2h - 3|\, [2 + h] = 2, V = 1.2 = 2$

$L = \lim_{h \to 0} |\, 4 - 2h - 3|\, [2 - h] = 1, R \neq L$, $\therefore$ not continuous

At $x = 1, R = \lim |\, 2 + 2h - 3|\, [1 + h] = 1.1 = 1,$

$V = |-1|\, [1] = 1$

$L = \lim_{h \to 0} \sin \frac{\pi}{2}(1 - h) = 1$

Since $R = L = V$ $\therefore$ continuous at $x = 1$.

R.H.D. $= \lim_{h \to 0} \frac{|\, 2 + 2h - 3|\, [1 + h] - 1}{h} = \lim_{h \to 0} \frac{|-1|\,.1 - 1}{h} = \lim_{h \to 0} \frac{1 - 1}{h} = 0$

L.H.D. $= \lim_{h \to 0} \frac{|\, 2 - 2h - 3|\, [1 - h] - 1}{-h} = \lim_{h \to 0} \frac{1.0 - 1}{-h} = \lim_{h \to 0} \frac{1}{h} = \infty$

Since R.H.D. $\neq$ L.H.D. $\therefore$ not differentiable. at $x = 1$.

2.4.2 Differentiability in an Open Interval

A function $f(x)$ defined in an open interval (a, b) is said to be differentiable or derivable in open interval (a, b) if it is differentiable at each point of (a, b).

Differentiability in a closed interval : A function $f : [a, b] \to R$ is said to be differentiable in [a, b] if

(1) $f'(x)$ exists for every x such that $a < x < b$ *i.e. f* is differentiable in (a, b).

(2) Right hand derivative of f at $x = a$ exists.

(3) Left hand derivative of f at $x = b$ exists.

Everywhere differentiable function : If a function is differentiable at each $x \in R$, then it is said to be everywhere differentiable. *e.g.*, A constant function, a polynomial function, $\sin x, \cos x$ etc. are everywhere differentiable.

Some standard results on differentiability

(1) Every polynomial function is differentiable at each $x \in R$.

(2) The exponential function $a^x, a > 0$ is differentiable at each $x \in R$.

(3) Every constant function is differentiable at each $x \in R$.

(4) The logarithmic function is differentiable at each point in its domain.

(5) Trigonometric and inverse trigonometric functions are differentiable in their domains.

(6) The sum, difference, product and quotient of two differentiable functions is differentiable.

(7) The composition of differentiable function is a differentiable function.

Important Tips

☞ *If f is derivable in the open interval (a, b) and also at the end points 'a' and 'b', then f is said to be derivable in the closed interval [a, b].*

☞ *A function f is said to be a differentiable function if it is differentiable at every point of its domain.*

☞ *If a function is differentiable at a point, then it is continuous also at that point.*

i.e. Differentiability ⇒ Continuity, but the converse need not be true.

☞ *If a function 'f' is not differentiable but is continuous at x = a, it geometrically implies a sharp corner or kink at x = a.*

☞ *If f(x) is differentiable at x = a and g(x) is not differentiable at x =a, then the product function f(x).g(x) can still be differentiable at x = a.*

☞ *If f(x) and g(x) both are not differentiable at x = a then the product function f(x).g(x) can still be differentiable at x = a.*

☞ *If f(x) is differentiable at x = a and g(x) is not differentiable at x = a then the sum function f(x) + g(x) is also not differentiable at x = a*

☞ *If f(x) and g(x) both are not differentiable at x = a, then the sum function may be a differentiable function.*

Example: 12 The set of points where the function $f(x)=\sqrt{1-e^{-x^2}}$ is differentiable

(a) $(-\infty,\infty)$ (b) $(-\infty,0)\cup(0,\infty)$ (c) $(-1,\infty)$ (d) None of these

Solution: (b) Clearly, $f(x)$ is differentiable for all non-zero values of x, For $x\neq 0$, we have $f'(x)=\dfrac{xe^{-x^2}}{\sqrt{1-e^{-x^2}}}$

Now, (L.H.D. at x = 0)

$$=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{\sqrt{1-e^{-h^2}}}{-h}=\lim_{h\to 0}-\frac{\sqrt{1-e^{-h^2}}}{h}=-\lim_{h\to 0}\sqrt{\frac{e^{h^2}-1}{h^2}}\times\frac{1}{\sqrt{e^{h^2}}}=-1$$

and, (RHD at x = 0) = $\lim_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0}=\lim_{h\to 0}\dfrac{\sqrt{1-e^{-h^2}}-0}{h}=\lim_{h\to 0}\sqrt{\dfrac{e^{h^2}-1}{h^2}}\times\dfrac{1}{\sqrt{e^{h^2}}}=1$.

So, $f(x)$ is not differentiable at $x=0$, Hence, the points of differentiability of $f(x)$ are $(-\infty,0)\cup(0,\infty)$.

Example: 13 The function $f(x)=e^{-|x|}$ is

(a) Continuous everywhere but not differentiable at $x=0$

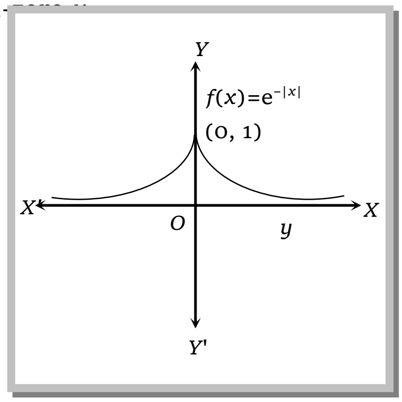
(b) Continuous and differentiable everywhere

(c) Not continuous at $x=0$

(d) None of these

Solution: (a) We have, $f(x)=\begin{cases}e^{-x}, & x\geq 0\\ e^{x}, & x<0\end{cases}$

Clearly, $f(x)$ is continuous and differentiable for all non-zero x.



Now, $\lim_{x\to 0^-}f(x)=\lim_{x\to 0}e^{x}=1$ and $\lim_{x\to 0^+}f(x)=\lim_{x\to 0}e^{-x}=1$

Also, $f(0)=e^{0}=1$

So, $f(x)$ is continuous for all x.

(LHD at $x=0$) = $\left(\dfrac{d}{dx}(e^{x})\right)_{x=0}=[e^{x}]_{x=0}=e^{0}=1$

(RHD at $x=0$) = $\left(\dfrac{d}{dx}(e^{-x})\right)_{x=0}=[-e^{-x}]_{x=0}=-1$

So, $f(x)$ is not differentiable at $x=0$.

Hence, $f(x)=e^{-|x|}$ is everywhere continuous but not differentiable at $x=0$. This fact is also evident from the graph of the function.

Example: 14 If $f(x)=\sqrt{1-\sqrt{1-x^2}}$, then $f(x)$ is

(a) Continuous on [–1, 1] and differentiable on (–1, 1) (b) Continuous on [–1,1] and differentiable on (–1, 0) ∪ (0, 1)

(c) Continuous and differentiable on [–1, 1] (d) None of these

Solution: (b) We have, $f(x)=\sqrt{1-\sqrt{1-x^2}}$. The domain of definition of $f(x)$ is [–1, 1].

For $x\neq 0, x\neq 1,\ x\neq -1$ we have $f'(x)=\frac{1}{\sqrt{1-\sqrt{1-x^2}}}\times\frac{x}{\sqrt{1-x^2}}$

Since $f(x)$ is not defined on the right side of $x=1$ and on the left side of $x=-1$. Also, $f'(x)\to\infty$ when $x\to -1^+$ or $x\to 1^-$. So, we check the differentiability at x = 0.

Now, (LHD at $x=0$) = $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$

$$=\lim_{h\to 0}\frac{\sqrt{1-\sqrt{1-h^2}}-0}{-h}=-\lim_{h\to 0}\frac{\sqrt{1-\{1-(1/2)h^2+(3/8)h^4+....\}}}{h}=-\lim_{h\to 0}\sqrt{\frac{1}{2}-\frac{3}{8}h^2+.....}=-\frac{1}{\sqrt{2}}$$

Similarly, (RHD at x = 0) = $\frac{1}{\sqrt{2}}$

Hence, $f(x)$ is not differentiable at x = 0.

Example: 15 Let $f(x)$ be a function differentiable at $x=c$. Then $\lim_{x\to c} f(x)$ equals

(a) $f'(c)$ (b) $f''(c)$ (c) $\frac{1}{f(c)}$ (d) None of these

Solution: (d) Since $f(x)$ is differentiable at $x=c$, therefore it is continuous at $x=c$. Hence, $\lim_{x\to c} f(x)=f(c)$.

Example: 16 The function $f(x)=(x^2-1)|x^2-3x+2|+\cos(|x|)$ is not differentiable at **[IIT Screening 1999]**

(a) – 1 (b) 0 (c) 1 (d) 2

Solution: (d) $(x^2-3x+2)=(x-1)(x-2)=+ive$

When $x<1$ or >2, $-ive$ when $1\le x\le 2$

Also $\cos|x|=\cos x$ (since $\cos(-x)=\cos x$)

$\therefore\ f(x)=-(x^2-1)(x^2-3x+2)+\cos x,\ \ 1\le x\le 2$

$\therefore\ f(x)=(x^2-1)(x^2-3x+2)+\cos x,\ \ x>2$ ....(i)

Evidently $f(x)$ is not differentiable at $x=2$ as $L'\neq R'$

Note : ❒ For all other values like $x<0,\ 0\le x<1,\ f(x)$ is same as given by (i).

Example: 17 If $f(x)=\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x\neq 0\\ 0, & x=0\end{cases}$, then $f(x)$ is **[AIEEE 2003]**

(a) Continuous as well as differentiable for all x (b) Continuous for all x but not differentiable at $x=0$

(c) Neither differentiable nor continuous at $x=0$ (d) Discontinuous every where

Solution: (b) $f(0)=0$ and $f(x)=xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}$

R.H.L. = $\lim_{h\to 0}(0+h)e^{-2/h}=\lim_{h\to 0}\frac{h}{e^{2/h}}=0$

L.H.L. = $\lim_{h\to 0}(0-h)e^{-\left(\frac{1}{h}-\frac{1}{h}\right)}=0$

$\therefore\ f(x)$ is continuous.

$$Rf'(x) \text{ at } (x=0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{he^{-2/h}}{h} = e^{-\infty} = 0$$

$$Lf'(x) \text{ at } (x=0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{-he^{-\left(\frac{1}{h}-\frac{1}{h}\right)}}{-h} = +1 \Rightarrow Lf'(x) \neq Rf'(x)$$

$f(x)$ is not differentiable at $x=0$.

Example: 18 The function $f(x) = x^2 \sin\frac{1}{x}, x\neq 0,\ f(0)=0$ at $x=0$ **[MP PET 2003]**

(a) Is continuous but not differentiable (b) Is discontinuous

(c) Is having continuous derivative (d) Is continuous and differentiable

Solution: (d) $\lim_{x\to 0} f(x) = x^2 \sin\left(\frac{1}{x}\right)$ but $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ and $x\to 0$

$\therefore \lim_{x\to 0^+} f(x) = 0 = \lim_{x\to 0^-} f(x) = f(0)$

Therefore $f(x)$ is continuous at $x=0$. Also, the function $f(x) = x^2\sin\frac{1}{x}$ is differentiable because

$$Rf'(x) = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h}-0}{h} = 0,\ Lf'(x) = \lim_{h\to 0}\frac{h^2\sin\left(\frac{1}{-h}\right)}{-h} = 0.$$

Example: 19 Which of the following is not true

(a) A polynomial function is always continuous (b) A continuous function is always differentiable

(c) A differentiable function is always continuous (d) e^x is continuous for all x

Solution: (b) A continuous function may or may not be differentiable. So (b) is not true.

Example: 20 If $f(x) = \operatorname{sgn}(x^3)$, then **[DCE 2001]**

(a) f is continuous but not derivable at $x=0$ (b) $f'(0^+) = 2$

(c) $f'(0^-) = 1$ (d) f is not derivable at $x=0$

Solution: (d) Here, $f(x) = \operatorname{sgn} x^3 = \begin{cases}\begin{cases}\frac{x^3}{|x^3|} & \text{for } x^3\neq 0\\ 0 & \text{for } x^3 = 0\end{cases}\\ \begin{cases}\frac{x}{|x|} & \text{for } x\neq 0\\ 0 & \text{for } x=0\end{cases}\\ \begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}\end{cases}$. Thus, $f(x) = \operatorname{sgn} x^3 = \operatorname{sgn} x$, which is neither continuous nor derivable at 0.

Note: ❒ $f'(0^+) = \lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+}\frac{1-0}{h}\to\infty$ and $f'(0^-) = \lim_{h\to 0^-}\frac{f(0-h)-f(0)}{h} = \lim_{h\to 0^-}\frac{-1-0}{h}\to\infty$.

$\therefore f'(0^+)\neq f'(0^-)$, $\therefore f$ is not derivable at $x=0$.

Example: 21 A function $f(x) = \begin{cases}1+x, & x\le 2\\ 5-x, & x>2\end{cases}$ is **[AMU 2001]**

(a) Not continuous at $x=2$ (b) Differentiable at $x=2$

(c) Continuous but not differentiable at $x=2$ (d) None of the above

Solution: (c) $\lim_{h\to 0^-} 1+(2-h)=3$, $\lim_{h\to 0^+} 5-(2+h)=3$, $f(2)=3$

Hence, f is continuous at $x=2$

Now $Rf'(x)=\lim_{h\to 0}\frac{5-(2+h)-3}{h}=-1$

$Lf'(x)=\lim_{h\to 0}\frac{1+(2-h)-3}{-h}=1$

$\because Rf'(x)\neq Lf'(x)$

$\therefore f$ is not differentiable at $x=2$.

Example: 22 Let $f:R\to R$ be a function. Define $g:R\to R$ by $g(x)=|f(x)|$ for all x. Then g is

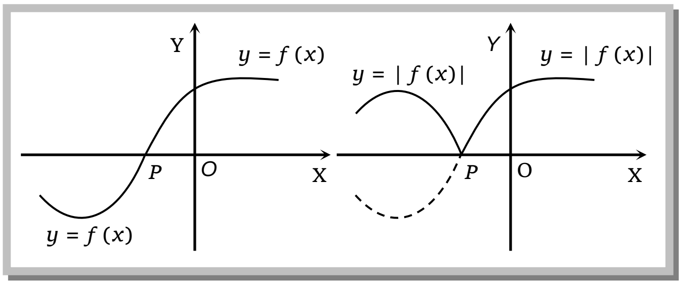
(a) Onto if f is onto

(b) One-one if f is one-one

(c) Continuous if f is continuous

(d) Differentiable if f is differentiable

Solution: (c) $g(x)=|f(x)|\geq 0$. So $g(x)$ cannot be onto. If $f(x)$ is one-one and $f(x_1)=-f(x_2)$ then $g(x_1)=g(x_2)$. So, '$f(x)$ is one-one' does not ensure that $g(x)$ is one-one.



If $f(x)$ is continuous for $x\in R$, $|f(x)|$ is also continuous for $x\in R$. This is obvious from the following graphical consideration.

So the answer (c) is correct. The fourth answer (d) is not correct from the above graphs $y=f(x)$ is differentiable at P while $y=|f(x)|$ has two tangents at P, *i.e.* not differentiable at P.